Model Theory of Gödel Logic

J. P. Aguilera

April 27, 2024

Gödel logics are real-valued logics intermediate between intuitionistic logic and classical logic; they are one of the three fundamental fuzzy logics. Although heavily studied from the point of view of proof theory and computer science, its model theory is relatively unexplored and quite remarkable. Many classical theorems about first-order logic, such as the Löwenheim-Skolem theorem or the Compactness theorem fail for first-order Gödel logics in their usual formulation. However, they can be resurrected in weaker forms if one replaces the role of "infinity" in these theorems by "having very large cardinality."

Theorem 0.1. *Let ϕ be a first-order sentence. The following are equivalent:*

1. *In real-valued Gödel logic, ϕ has a model of every infinite cardinality.*
2. *In real-valued Gödel logic, ϕ has a model of cardinality $\beth_\omega = \sup\{\aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}, \dots\}$.*

Moreover, the equivalence fails if one replaces $\beth_\omega$ by any smaller cardinality.

Theorem 0.2. *Let Γ be a set of sentences in first-order logic and let κ be the smallest ω_1-strongly compact cardinal. Suppose that every $\Gamma' \subset \Gamma$ of cardinality $< \kappa$ has a model in Gödel logic. Then, Γ has a model in Gödel logic.*

Moreover, this is not true if one replaces κ by a smaller cardinality.

While for classical logic one can replace $\beth_\omega$ and κ, respectively, by $\aleph_0$ in the two theorems above, this is no longer possible for Gödel logic. Note that ω_1-strongly compact cardinals are so large that they cannot be proved to exist within the Zermelo-Fraenkel axioms.

While these interactions between model theory and set theory are usually associated with logics much stronger than first-order logic, it is quite remarkable that they arise in the context of logics weaker than classical (finitary) first-order logic as well.